Reference Answer of Lecture 4

1.

$$\begin{cases} x' = f(t,x,\mu) \\ x(t_0) = x_0 \end{cases} \quad (E_\mu) \qquad \begin{cases} x' = f(t,x,\mu), & \mu'=0 \\ x(t_0) = x_0 & \mu(t_0)=\mu \end{cases} \quad (H_\mu)$$

a) $z(t) = (x(t), \mu(t))$ is a solution of $(H_\mu) \Leftrightarrow x(t)$ is a solution of (E_μ) and $\mu(t) \equiv \mu$;

b)If (E_μ) has a unique solution,so does (H_μ)

Proof:

a)" $\Rightarrow$ " $\begin{cases} x' = f(t,x,\mu), & \mu'=0 \\ x(t_0) = x_0 & \mu(t_0)=\mu \end{cases}$ firstly means $x(t)$ is a solution of (E_μ) ,besides

$\mu'=0$ means μ is a constant.And $\mu(t_0)=\mu$ means $\mu(t) \equiv \mu$

" $\Leftarrow$ " $x(t)$ is a solution of (E_μ) means $\begin{cases} x' = f(t,x,\mu), \\ x(t_0) = x_0 \end{cases}$,

then $\mu(t) \equiv \mu$ means $\mu'=0$ and $\mu(t_0)=\mu$ .So $z(t) = (x(t), \mu(t))$ is a solution of (H_μ)

b) If (E_μ) has a unique solution,but (H_μ) has two solutions.Suppose $z_1(t) = (x_1(t), \mu_1(t))$

$z_2(t) = (x_2(t), \mu_2(t))$ are both solutions of (H_μ) ,then

$$\begin{cases} x_1' = f(t,x_1,\mu_1), & \mu_1'=0 \\ x_1(t_0) = x_0 & \mu_1(t_0)=\mu \end{cases} \text{ and } \begin{cases} x_2' = f(t,x_2,\mu_2), & \mu_2'=0 \\ x_2(t_0) = x_0 & \mu_2(t_0)=\mu \end{cases}$$

$\mu_i'=0$ and $\mu_i(t_0)=\mu$ means $\mu_1 \equiv \mu_2 \equiv \mu$

(E_μ) has a unique solution means $x_1 \equiv x_2 \Rightarrow z_1 \equiv z_2$,so (H_μ) has a unique solution.

2. $Y_{n+1}(t,t_0,x_0) = I + \int_{t_0}^{t} f_x'(s, x_n(s,t_0,x_0))Y_n(s,t_0,x_0)ds$ .Show that $\{Y_n(t,t_0,x_0)\}$ is uniformly convergent to $Y(t,t_0,x_0)$ that is continuous on $(t,t_0,x_0) \in G$.

Proof: Suppose $\|Y_n(s,t_0,x_0)\| \le N$, $\|f_x'(t, x_n(t,t_0,x_0))\| \le M'$, $f_x'(t, x_n(t,t_0,x_0))$ satisfies Liptischz inequality $\|f_x'(t, x(t,t_0,x_0)) - f_x'(t, y(t,t_0,x_0))\| \le L'\|x-y\|$

$$\|Y_1 - Y_0\| = \left\| \int_{t_0}^{t} f_x'(s, x_0(s,t_0,x_0))Y_0(s,t_0,x_0)ds \right\| \le M'(t-t_0)$$

$$\|Y_2 - Y_1\| = \left\| \int_{t_0}^{t} f_x'(s, x_1(s,t_0,x_0))Y_1(s,t_0,x_0)ds - \int_{t_0}^{t} f_x'(s, x_0(s,t_0,x_0))Y_0(s,t_0,x_0)ds \right\|$$

$$\leq \int_{t_0}^{t} \left\| f_x'(s, x_1(s,t_0,x_0)) - f_x'(s, x_0(s,t_0,x_0)) \right\| \left\| \mathrm{Y}_1(s,t_0,x_0) \right\| ds$$

$$+ \int_{t_0}^{t} \left\| f_x'(s, x_0(s,t_0,x_0)) \right\| \left\| \mathrm{Y}_1(s,t_0,x_0) - \mathrm{Y}_0(s,t_0,x_0) \right\| ds$$

$$\leq \mathrm{L'N} \frac{\mathrm{ML}}{2!} |t-t_0|^2 + (M')^2 \frac{|t-t_0|^2}{2!}$$

$$\left\| \mathrm{Y}_3 - \mathrm{Y}_2 \right\| = \left\| \int_{t_0}^{t} f_x'(s, x_2(s,t_0,x_0)) \mathrm{Y}_2(s,t_0,x_0) ds - \int_{t_0}^{t} f_x'(s, x_1(s,t_0,x_0)) \mathrm{Y}_1(s,t_0,x_0) ds \right\|$$

$$\leq \int_{t_0}^{t} \left\| f_x'(s, x_2(s,t_0,x_0)) - f_x'(s, x_1(s,t_0,x_0)) \right\| \left\| \mathrm{Y}_2(s,t_0,x_0) \right\| ds$$

$$+ \int_{t_0}^{t} \left\| f_x'(s, x_0(s,t_0,x_0)) \right\| \left\| \mathrm{Y}_2(s,t_0,x_1) - \mathrm{Y}_1(s,t_0,x_0) \right\| ds$$

$$\leq (\mathrm{L'N})^2 \frac{\mathrm{ML}^2}{3!} |t-t_0|^3 + (M')^3 \frac{|t-t_0|^3}{3!}$$

$\vdots$

$$\left\| \mathrm{Y}_n - \mathrm{Y}_{n-1} \right\| \leq (\mathrm{L'N})^{n-1} \frac{\mathrm{ML}^{n-1}}{\mathrm{n!}} |t-t_0|^n + (M')^n \frac{|t-t_0|^n}{\mathrm{n!}}$$

Since the series $\sum_{n=1}^{\infty} (\mathrm{L'N})^{n-1} \frac{\mathrm{ML}^{n-1}}{\mathrm{n!}} |t-t_0|^n + \sum_{n=1}^{\infty} (M')^n \frac{|t-t_0|^n}{\mathrm{n!}}$ is convergent on $(t,t_0,x_0) \in G$

So does $\mathrm{Y}_0(t) + \sum_{n=1}^{\infty} [\mathrm{Y}_n(t) - \mathrm{Y}_{n-1}(t)]$

i.e. $\{\mathrm{Y}_n(t,t_0,x_0)\}$ is uniformly convergent to $\mathrm{Y}(t,t_0,x_0)$ that is continuous on $(t,t_0,x_0) \in G$

3.It's similarly proved like problem2.

4.If $x' = f(t,x)$,where $f(t,x)$ is continuous and $\|f(t,x)\| \leq M\|x\|$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$,

Show that for any (t_0, x_0) ,the solution $x(t)$ has $\mathrm{I}_{\max} = (-\infty, \infty)$.

Proof:

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds$$
$$\|x(t)\| \leq \|x_0\| + M \int_{t_0}^{t} \|x(s)\| ds ,$$

From Gronwall's inequality we get

$$\|x(t)\| \leq \|x_0\| e^{M(t-t_0)}$$

Let $t \to \omega_+$,if $\omega_+ < +\infty \Rightarrow \|x(t)\| < +\infty$ contradicts with Continuation theorem.So $\omega_+ = \infty$.

Similarly $\omega_- = -\infty$.

5.All solutions of the Riccati equation $x' = 1 + x^2$ have a finite escape.

Proof: $Let\ \omega_+ > 0$, $x'(t) = 1 + x^2(t)$.

$$\frac{dx(t)}{1+x^2(t)} = dt \Rightarrow \arctan x(t) - \arctan x(t_0) = t - t_0 \quad \forall t \in [t_0, \omega_+]$$

$$\therefore 0 \le t - t_0 \le \pi \Rightarrow 0 \le \omega_+ \le \pi$$

This completes the proof.